

A STATEMENT OF THE FUNDAMENTAL LEMMA

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ABSTRACT. These notes give a statement of the fundamental lemma, which is a conjectural identity between p -adic integrals.

1. INTRODUCTION

Notation. Let F be a p -adic field, given either as a finite field extension of \mathbb{Q}_p , or as the field $F = \mathbb{F}_q((t))$. Let \mathbb{F}_q (a finite field with q elements and characteristic p) be the residue field of F . Let \bar{F} be a fixed algebraic closure of F . Let F^{un} be the maximal unramified extension of F in \bar{F} . For simplicity, we also assume that the characteristic of F is not 2.

The fundamental lemma pertains to groups that satisfy a series of hypotheses. Here is the first.

Assumption 1.1. *G is a connected reductive linear algebraic group that is defined over F .*

The following examples give the F -points of three different families of connected reductive linear algebraic groups: orthogonal, symplectic, and unitary groups.

Example 1.2. Let $M(n, F)$ be the algebra of n by n matrices with coefficients in F . Let $J \in M(n, F)$ be a symmetric matrix with nonzero determinant. The special orthogonal group with respect to the matrix J is

$$\mathrm{SO}(n, J, F) = \{X \in M(n, F) \mid {}^tXJX = J, \quad \det(X) = 1\}.$$

Example 1.3. Let $J \in M(n, F)$, with $n = 2k$, be a skew-symmetric matrix ${}^tJ = -J$ with nonzero determinant. The symplectic group with respect to J is defined in a similar manner:

$$\mathrm{Sp}(2k, J, F) = \{X \in M(2k, F) \mid {}^tXJX = J\}.$$

Example 1.4. Let E/F be a separable quadratic extension. Let \bar{x} be the Galois conjugate of $x \in E$ with respect to the nontrivial automorphism of E fixing F . For any $A \in M(n, E)$, let \bar{A} be the matrix obtained by taking the Galois conjugate of each coefficient of A . Let $J \in M(n, E)$ satisfy ${}^t\bar{J} = J$ and have a nonzero determinant. The unitary group with respect to J and E/F is

$$U(n, J, F) = \{X \in M(n, E) \mid {}^t\bar{X}JX = J\}.$$

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The algebraic groups $SO(n, J)$, $Sp(2k, J)$, and $U(n, J)$ satisfy Assumption 1.1.

Assumption 1.5. *G splits over an unramified field extension.*

That is, there is an unramified extension F_1/F such that $G \times_F F_1$ is split.

Example 1.6. In the first two examples above (orthogonal and symplectic), if we take J to have the special form

$$(1.6.1) \quad J = \begin{pmatrix} 0 & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix}$$

(that is, nonzero entries from F along the cross-diagonal and zeros elsewhere), then G splits over F . In the third example (unitary), if J has this same form and if E/F is unramified, then the unitary group splits over the unramified extension E of F .

Assumption 1.7. *G is quasi-split.*

This means that there is an F -subgroup $B \subset G$ such that $B \times_F \bar{F}$ is a Borel subgroup of $G \times_F \bar{F}$.

Example 1.8. In all three cases (orthogonal, symplectic, and unitary), if J has the cross-diagonal form 1.6.1, then G is quasi-split. In fact, we can take the points of B to be the set of upper triangular matrices in $G(F)$.

Assumption 1.9. *K is a hyperspecial maximal compact subgroup of $G(F)$, in the sense of Definition 1.11.*

Example 1.10. Let O_F be the ring of integers of F and let $K = GL(n, O_F)$. This is a hyperspecial maximal compact subgroup of $GL(n, F)$.

Definition 1.11. K is *hyperspecial* if there exists \mathcal{G} such that the following conditions are satisfied.

- \mathcal{G} is a smooth group scheme over O_F ,
- $G = \mathcal{G} \times_{O_F} F$,
- $\mathcal{G} \times_{O_F} \mathbb{F}_q$ is reductive,
- $K = \mathcal{G}(O_F)$.

Example 1.12. In all three examples (orthogonal, symplectic, and unitary), take G to have the form of Example 1.6. Assume that each cross-diagonal entry is a *unit* in the ring of integers. Assume further that the residual characteristic is not 2. Then the equations

$${}^t X J X = J \quad (\text{or in the unitary case } {}^t \bar{X} J X = J)$$

define a group scheme \mathcal{G} over O_F , and $\mathcal{G}(O_F)$ is hyperspecial.

2. CLASSIFICATION OF UNRAMIFIED REDUCTIVE GROUPS

Definition 2.1. If G is quasi-split and splits over an unramified extension (that is, if G satisfies Assumptions 1.5 and 1.7), then G is said to be an *unramified reductive group*.

Let G be an unramified reductive group. It is classified by data (called root data)

$$(X^*, X_*, \Phi, \Phi^\vee, \sigma).$$

The data is as follows:

- X^* is the character group of a Cartan subgroup of G .
 - X_* is the cocharacter group of the Cartan subgroup.
 - $\Phi \subset X^*$ is the set of roots.
 - $\Phi^\vee \subset X_*$ is the set of coroots.
 - σ is an automorphism of finite order of X^* sending a set of simple roots in Φ to itself.
- σ is obtained from the action on the character group induced from the Frobenius automorphism of $\text{Gal}(F^{un}/F)$ on the maximally split Cartan subgroup in G .

The first four elements $(X^*, X_*, \Phi, \Phi^\vee)$ classify split reductive groups G over F . For such groups $\sigma = 1$.

3. ENDOSCOPIC GROUPS

H is an unramified endoscopic group of G if it is an unramified reductive group over F whose classifying data has the form

$$(X^*, X_*, \Phi_H, \Phi_H^\vee, \sigma_H).$$

The first two entries are the same for G as for H . To distinguish the data for H from that for G , we add subscripts H or G , as needed. The data for H is subject to the constraints that there exists an element $s \in \text{Hom}(X_*, \mathbb{C}^\times)$ and a Weyl group element $w \in W(\Phi_G)$ such that

- $\Phi_H^\vee = \{\alpha \in \Phi_G^\vee \mid s(\alpha) = 1\}$,
- $\sigma_H = w \circ \sigma_G$, and
- $\sigma_H(s) = s$.

3.1. Endoscopic groups for $SL(2)$. As an example, we determine the unramified endoscopic groups of $G = SL(2)$. The character group X^* can be identified with \mathbb{Z} , where $n \in \mathbb{Z}$ is identified with the character on the diagonal torus given by

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto t^n.$$

The set Φ can be identified with the subset $\{\pm 2\}$ of \mathbb{Z} :

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto t^{\pm 2}.$$

The cocharacter group X_* is also identified with \mathbb{Z} , where $n \in \mathbb{Z}$ is identified with

$$t \mapsto \begin{pmatrix} t^n & 0 \\ 0 & t^{-n} \end{pmatrix}.$$

Under this identification $\Phi^\vee = \{\pm 1\}$. Since the group is split, $\sigma = 1$.

We get an unramified endoscopic group by selecting $s \in \text{Hom}(X_*, \mathbb{C}^\times) \cong \mathbb{C}^\times$ and $w \in W(\Phi)$.

$$(3.0.1) \quad \begin{aligned} \Phi_H^\vee = \{\alpha \mid s(\alpha) = 1\} &= \{n \in \{\pm 1\} \mid s^n = 1\} \\ &= \text{if } (s = 1) \text{ then } \Phi_G^\vee \text{ else } \emptyset. \end{aligned}$$

We consider two cases, according as w is nontrivial or trivial. If w is the nontrivial reflection, then $\sigma_H = w$ acts by negation on \mathbb{Z} . Thus,

$$(\sigma_H(s) = s) \implies (s^{-1} = s) \implies (s = \pm 1).$$

If $s = 1$, then σ_H does not fix a set of simple roots as required. So $s = -1$ and $\Phi_H^\vee = \emptyset$. Thus, the root data of H is

$$(\mathbb{Z}, \mathbb{Z}, \emptyset, \emptyset, w)$$

This determines H up to isomorphism as $H = U_E(1)$, a 1-dimensional torus split by an unramified quadratic extension E/F .

If w is trivial, then there are two further cases, according as Φ_H is empty or not:

- The endoscopic group \mathbb{G}_m has root data

$$(\mathbb{Z}, \mathbb{Z}, \emptyset, \emptyset, 1).$$

- The endoscopic group $H = SL(2)$ has root data

$$(\mathbb{Z}, \mathbb{Z}, \{\pm 2\}, \{\pm 1\}, 1).$$

In summary, the three unramified endoscopic groups of $SL(2)$ are $U_E(1)$, \mathbb{G}_m , and $SL(2)$ itself.

3.2. Endoscopic groups for $PGL(2)$. As a second complete example, we determine the endoscopic groups of $PGL(2)$. The group $PGL(2)$ is dual to $SL(2)$ in the sense that the coroots of one group can be identified with the roots of the other group. The root data for $PGL(2)$ is

$$(\mathbb{Z}, \mathbb{Z}, \{\pm 1\}, \{\pm 2\}, 1).$$

When the Weyl group element is trivial, then the calculation is almost identical to the calculation for $SL(2)$. We find that there are again two cases, according as Φ_H is empty or not:

- The endoscopic group \mathbb{G}_m has root data

$$(\mathbb{Z}, \mathbb{Z}, \emptyset, \emptyset, 1).$$

- The endoscopic group $H = PGL(2)$ has root data

$$(\mathbb{Z}, \mathbb{Z}, \{\pm 1\}, \{\pm 2\}, 1).$$

When the Weyl group element w is nontrivial, then $s \in \{\pm 1\}$, as in the $SL(2)$ calculation.

$$(3.0.2) \quad \Phi_H^\vee = \{\alpha \mid s(\alpha) = 1\} = \{n \in \{\pm 2\} \mid s^n = 1\} = \Phi_G^\vee.$$

From this, we see that picking w to be nontrivial is incompatible with the requirement that $\sigma_H = w$ must fix a set of simple roots. Thus, there are no endoscopic groups with w nontrivial.

In summary, the two endoscopic groups of $PGL(2)$ are \mathbb{G}_m and $PGL(2)$ itself.

3.3. Elliptic Endoscopic groups.

Definition 3.1. An unramified endoscopic group H is said to be *elliptic*, if

$$(\mathbb{R}\Phi_G)^{W(\Phi_H) \rtimes \langle \sigma_H \rangle} = (0).$$

That is, the span of the set of roots of G has no invariant vectors under the Weyl group of H and the automorphism σ_H .

The origin of the term *elliptic* is the following. We will see below that each Cartan subgroup of H is isomorphic to a Cartan subgroup of G . (Here and elsewhere, when we speak of an isomorphism between algebraic groups defined over F , we mean an isomorphism over F .) The condition on H for it to be elliptic is precisely the condition that is needed for some Cartan subgroup of H to be isomorphic to an *elliptic* Cartan subgroup of G .

Example 3.2. We calculate the elliptic unramified endoscopic subgroups of $SL(2)$. We may identify $\mathbb{R}\Phi$ with $\mathbb{R}\{\pm 2\}$ and hence with \mathbb{R} . An unramified endoscopic group is elliptic precisely when $W(\Phi_H)$ or $\langle \sigma_H \rangle$ contains the nontrivial reflection $x \mapsto -x$. When $H = SL(2)$, the Weyl group contains the nontrivial reflection. When $H = U_E(1)$, the element σ_H is the nontrivial reflection. But when $H = \mathbb{G}_m$, both $W(\Phi_H)$ and $\langle \sigma_H \rangle$ are trivial. Thus, $H = SL(2)$ and $H = U_E(1)$ are elliptic, but $H = \mathbb{G}_m$ is not.

3.4. An exercise: elliptic endoscopic groups of unitary groups.

This exercise is a calculation of the elliptic unramified endoscopic groups of $U(n, J)$. We assume that J is a cross-diagonal matrix with units along the cross-diagonal as in Section 1.6.1. We give a few facts about the endoscopic groups of $U(n, J)$ and leave it as an exercise to fill in the details.

Let $T = \{\text{diag}(t_1, \dots, t_n)\}$ be the group of diagonal n by n matrices. The character group X^* can be identified with \mathbb{Z}^n in such a way that the character

$$\text{diag}(t_1, \dots, t_n) \mapsto t_1^{k_1} \cdots t_n^{k_n}$$

is identified with (k_1, \dots, k_n) .

The cocharacter group can be identified with \mathbb{Z}^n in such a way that the cocharacter

$$t \mapsto \text{diag}(t^{k_1}, \dots, t^{k_n})$$

is identified with (k_1, \dots, k_n) .

Let e_i be the basis vector of \mathbb{Z}^n whose j -th entry is Kronecker δ_{ij} . The set of roots can be identified with

$$\Phi = \{e_i - e_j \mid i \neq j\}.$$

The set of coroots Φ^\vee can be identified with the set of roots Φ under the isomorphism $X_* \cong \mathbb{Z}^n \cong X^*$.

We may identify $\text{Hom}(X_*, \mathbb{C}^\times)$ with $\text{Hom}(\mathbb{Z}^n, \mathbb{C}^\times) = (\mathbb{C}^\times)^n$. Thus, we take the element s in the definition of endoscopic group to have the form $s = (s_1, \dots, s_n) \in (\mathbb{C}^\times)^n$. The element $\sigma = \sigma_G$ acts on characters and cocharacters by

$$\sigma(k_1, \dots, k_n) = (-k_n, \dots, -k_1).$$

Let $I = \{1, \dots, n\}$. Show that if H is an elliptic unramified endoscopic group, then there is a partition

$$I = I_1 \amalg I_2$$

with $s_i = 1$ for $i \in I_1$ and $s_i = -1$ otherwise. The elliptic endoscopic group is a product of two smaller unitary groups $H = U(n_1) \times U(n_2)$, where $n_i = \#I_i$, for $i = 1, 2$.

4. CARTAN SUBGROUPS

All unramified reductive groups are classified by their root data. This includes the classification of unramified tori T as a special case (in this case, the set of roots and the set of coroots are empty):

$$(X^*(T), X_*(T), \emptyset, \emptyset, \sigma).$$

We can extend this classification to ramified tori. If T is any torus over F , it is classified by

$$(X^*(T), X_*(T), \rho),$$

where ρ is now allowed to be any homomorphism

$$\rho : \text{Gal}(\bar{F}/F) \rightarrow \text{Aut}(X^*(T))$$

with finite image.

A basic fact is that T embeds over F as a Cartan subgroup in a given unramified reductive group G if and only if the following two conditions hold.

- The image of ρ in $\text{Aut}(X^*(T))$ is contained in $W(\Phi_G) \rtimes \langle \sigma_G \rangle$.
- There is a commutative diagram:

$$\begin{array}{ccc} \text{Gal}(\bar{F}/F) & \longrightarrow & \text{Gal}(F^{un}/F) \\ \rho \downarrow & & \downarrow \text{Frob} \mapsto \sigma_G \\ W(\Phi_G) \rtimes \langle \sigma_G \rangle & \xrightarrow{w \rtimes \tau \mapsto \tau} & \langle \sigma_G \rangle. \end{array}$$

It follows that every Cartan subgroup T_H of H is isomorphic over F with a Cartan subgroup T_G of G . (To check this, simply observe that these two

conditions are more restrictive for H than the corresponding conditions for G .) The isomorphism can be chosen to induce an isomorphism of Galois modules between the character group (and cocharacter group) of T_H and that of T_G .

We say that a semisimple element in a reductive group is *strongly regular*, if its centralizer is a Cartan subgroup. If $\gamma \in H(F)$ is strongly regular semisimple, then its centralizer T_H is isomorphic to some $T_G \subset G$. Let $\gamma_0 \in T_G(F) \subset G(F)$ be the element in $G(F)$ corresponding to $\gamma \in T_H(F) \subset H(F)$, under this isomorphism.

Remark 4.1. The element γ_0 is not uniquely determined by γ . The Cartan subgroup T_G can always be replaced with a conjugate $g^{-1}T_Gg$, $g \in G(F)$, without altering the root data. However, the non-uniqueness runs deeper than this. An example will be worked in Section 8.1 to show how to deal with the problem of non-uniqueness. Non-uniqueness of γ_0 is related to stable conjugacy, which is our next topic.

5. STABLE CONJUGACY

Definition 5.1. Let δ and δ' be strongly regular semisimple elements in $G(F)$. They are *conjugate* if $g^{-1}\delta g = \delta'$ for some $g \in G(F)$. They are *stably conjugate* if $g^{-1}\delta g = \delta'$ for some $g \in G(\bar{F})$.

Example 5.2. Let $G = SL(2)$ and $F = \mathbb{Q}_p$. Assume that $p \neq 2$ and that u is a unit that is not a square in \mathbb{Q}_p . Let $\epsilon = \sqrt{u}$ in an unramified quadratic extension of \mathbb{Q}_p . We have the matrix calculation

$$\begin{pmatrix} 1+p & 1 \\ 2p+p^2 & 1+p \end{pmatrix} \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} \begin{pmatrix} 1+p & u^{-1} \\ (2p+p^2)u & 1+p \end{pmatrix}.$$

This matrix calculation shows that the matrices

$$(5.2.1) \quad \begin{pmatrix} 1+p & 1 \\ 2p+p^2 & 1+p \end{pmatrix} \text{ and } \begin{pmatrix} 1+p & u^{-1} \\ (2p+p^2)u & 1+p \end{pmatrix}$$

of $SL(2, \mathbb{Q}_p)$ are stably conjugate. The diagonal matrix that conjugates one to the other has coefficients that lie in a quadratic extension. A short calculation shows that the matrices 5.2.1 are not conjugate by a matrix of $SL(2, \mathbb{Q}_p)$.

5.1. Cocycles. Let γ_0 and γ' be stably conjugate strongly regular semisimple elements of $G(F)$. We view γ_0 as a fixed base point and γ' as variable. If $\tau \in \text{Gal}(\bar{F}/F)$, then

$$(5.2.2) \quad \begin{aligned} g^{-1}\gamma_0 g &= \gamma', \text{ (with } g \in G(\bar{F}), \gamma_0, \gamma' \in G(F)) \\ \tau(g)^{-1}\tau(\gamma_0)\tau(g) &= \tau(\gamma'), \\ \tau(g)^{-1}\gamma_0\tau(g) &= g^{-1}\gamma_0 g, \\ \gamma_0(\tau(g)g^{-1}) &= (\tau(g)g^{-1})\gamma_0, \\ \gamma_0 a_\tau &= a_\tau \gamma_0, \text{ with } a_\tau = \tau(g)g^{-1}. \end{aligned}$$

The element a_τ centralizes γ_0 and hence gives an element of the centralizer T . Viewed as a function of $\tau \in \text{Gal}(\bar{F}/F)$, a_τ satisfies the cocycle relation

$$\tau_1(a_{\tau_2})a_{\tau_1} = a_{\tau_1\tau_2}.$$

It is continuous in the sense that there exists a field extension F_1/F for which $a_\tau = 1$, for all $\tau \in \text{Gal}(\bar{F}/F_1)$. Thus, a_τ gives a class in

$$H^1(\text{Gal}(\bar{F}/F), T(\bar{F})),$$

which is defined to be the group of all continuous cocycles with values in T , modulo the subgroup of all continuous cocycles of the form

$$b_\tau = \tau(t)t^{-1},$$

for some $t \in T(\bar{F})$.

A general calculation of the group $H^1(\text{Gal}(\bar{F}/F), T)$ is achieved by the *Tate-Nakayama* isomorphism. Let F_1/F be a Galois extension that splits the Cartan subgroup T .

Theorem 5.3. (*Tate-Nakayama isomorphism* [27]) *The group $H^1(\text{Gal}(\bar{F}/F), T)$ is isomorphic to the quotient of the group*

$$\{u \in X_* \mid \sum_{\tau \in \text{Gal}(F_1/F)} \tau u = 0\}$$

by the subgroup generated by the set

$$\{u \in X_* \mid \exists \tau \in \text{Gal}(F_1/F) \exists v \in X_*. u = \tau v - v\}.$$

Example 5.4. Let $T = U_E(1)$ (the torus that made an appearance earlier as an endoscopic group of $SL(2)$). As was shown above, the group of cocharacters can be identified with \mathbb{Z} . The splitting field of T is the quadratic extension field E . The nontrivial element $\tau \in \text{Gal}(E/F)$ acts by reflection on $X_* \cong \mathbb{Z}$: $\tau(u) = -u$. By the Tate-Nakayama isomorphism, the group $H^1(\text{Gal}(\bar{F}/F), U_E(1))$ is isomorphic to

$$\{u \in \mathbb{Z} \mid u + \tau u = 0\} / \{u \in \mathbb{Z} \mid \exists v. u = \tau v - v\} = \mathbb{Z}/2\mathbb{Z}.$$

Let H be an unramified endoscopic group of G . Suppose that T_H is a Cartan subgroup of H . Let T_G be an isomorphic Cartan subgroup in G . The data defining H includes the existence of an element $s \in \text{Hom}(X_*, \mathbb{C}^\times)$; that is, a character of the abelian group X_* . Fix one such character s . We can restrict this character to get a character of

$$\{u \in X_* \mid \sum_{\tau \in \text{Gal}(F_1/F)} \tau u = 0\}.$$

It can be shown that the character s is trivial on

$$\{u \in X_* \mid \exists \tau \in \text{Gal}(F_1/F) \exists v \in X_*. u = \tau v - v\}.$$

Thus, by the Tate-Nakayama isomorphism, the character s determines a character κ of the cohomology group

$$H^1(\text{Gal}(\bar{F}/F), T).$$

In this way, each cocycle a_τ gives a complex constant $\kappa(a_\tau) \in \mathbb{C}^\times$.

Example 5.5. The element $s \in \mathbb{C}^\times$ giving the endoscopic group $H = U_E(1)$ of $SL(2)$ is $s = -1$, which may be identified with the character $n \mapsto (-1)^n$ of \mathbb{Z} . This gives the nontrivial character κ of

$$H^1(\text{Gal}(\bar{F}/F), U_E(1)) \cong \mathbb{Z}/2\mathbb{Z}.$$

6. STATEMENT OF THE FUNDAMENTAL LEMMA

6.1. Context. Let G be an unramified connected reductive group over F . Let H be an unramified endoscopic group of G . Let $\gamma \in H(F)$ be a strongly regular semisimple element. Let $T_H = C_H(\gamma)$, and let T_G be a Cartan subgroup of G that is isomorphic to it. More details will be given below about how to choose T_G . The choice of T_G matters! Let $\gamma \in T_H(F)$ map to $\gamma_0 \in T_G(F)$ under this isomorphism.

By construction, γ_0 is semisimple. However, as G may have more roots than H , it is possible for γ_0 to be singular, even when γ is strongly regular. If $\gamma \in H(F)$ is a strongly regular semisimple element with the property that γ_0 is also strongly regular, then we will call γ a *strongly G -regular* element of $H(F)$.

If γ' is stably conjugate to γ_0 with cocycle a_τ , then $s \in \text{Hom}(X_*, \mathbb{C}^\times)$ gives $\kappa(a_\tau) \in \mathbb{C}^\times$.

Let K_G and K_H be hyperspecial maximal compact subgroups of G and H . Let $\chi_{G,K}$ and $\chi_{H,K}$ be the characteristic functions of these hyperspecial subgroups. Set

$$(6.0.1) \quad \Lambda_{G,H}(\gamma) = \left(\prod_{\alpha \in \Phi_G} |\alpha(\gamma_0) - 1|^{1/2} \right) \left[\frac{\text{vol}(K_T, dt)}{\text{vol}(K, dg)} \right] \sum_{\gamma' \sim \gamma_0} \kappa(a_\tau) \int_{C_G(\gamma', F) \backslash G(F)} \chi_{G,K}(g^{-1} \gamma' g) \frac{dg}{dt'}.$$

The set of roots Φ_G are taken to be those relative to T_G . The sum runs over all stable conjugates γ' of γ_0 , up to conjugacy. This is a finite sum. The group K_T is defined to be the maximal compact subgroup of T_G . Equation 6.0.1 is a finite linear combination of orbital integrals (that is, integrals over conjugacy classes in the group with respect to an invariant measure). The Haar measures dt' on $C_G(\gamma', F)$ and dt on $T_G(F)$ are chosen so that stable conjugacy between the two groups is measure preserving. This particular linear combination of integrals is called a κ -orbital integral because of the term $\kappa(a_\tau)$ that gives the coefficients of the linear combination. Note that the integration takes place in the group G , and yet the parameter γ is an element of $H(F)$.

The volume terms $\text{vol}(K, dg)$ and $\text{vol}(K_T, dt)$ serve no purpose other than to make the entire expression independent of the choice of Haar measures dg and dt , which are only defined up to a scalar multiple.

We can form an analogous linear combination of orbital integrals on the group H . Set

(6.0.2)

$$\Lambda_H^{st}(\gamma) = \left(\prod_{\alpha \in \Phi_H} |\alpha(\gamma) - 1|^{1/2} \right) \left[\frac{\text{vol}(K_T, dt)}{\text{vol}(K_H, dh)} \right] \sum_{\gamma' \sim \gamma} \int_{C_H(\gamma', F) \backslash H(F)} \chi_{H,K}(h^{-1}\gamma'h) \frac{dh}{dt'}.$$

This linear combination of integrals is like $\Lambda_{G,H}(\gamma)$, except that H replaces G , K_H replaces K_G , Φ_H (taken relative to T_H) replaces Φ_G , and so forth. Also, the factor $\kappa(a_\tau)$ has been dropped. The linear combination of Equation 6.0.2 is called a stable orbital integral, because it extends over all stable conjugates of the element γ without the factor κ . The superscript *st* in the notation is for ‘stable.’

Conjecture 6.1. (*The fundamental lemma*) For every $\gamma \in H(F)$ that is strongly G -regular semisimple,

$$\Lambda_{G,H}(\gamma) = \Lambda_H^{st}(\gamma).$$

Remark 6.2. There have been serious efforts over the past twenty years to prove the fundamental lemma. These efforts have not yet led to a proof. Thus, the fundamental lemma is not a lemma; it is a conjecture with a misleading name. Its name leads one to speculate that the authors of the conjecture may have severely underestimated the difficulty of the conjecture.

Remark 6.3. Special cases of the fundamental lemma have been proved. The case $G = SL(n)$ was proved by Waldspurger [28]. Building on the work of [5], Laumon has proved that the fundamental lemma for $G = U(n)$ follows from a purity conjecture [21]. The fundamental lemma has not been proved for any other general families of groups. The fundamental lemma has been proved for some groups G of small rank, such as $SU(3)$ and $Sp(4)$. See [2], [7], [10].

6.2. The significance of the fundamental lemma. The Langlands program predicts correspondences $\pi \leftrightarrow \pi'$ between the representation theory of different reductive groups. There is a local program for the representation theory of reductive groups over locally compact fields, and a global program for *automorphic* representations of reductive groups over the adèle rings of global fields.

The Arthur-Selberg trace formula has emerged as a powerful tool in the Langlands program. In crude terms, one side of the trace formula contains terms related to the characters of automorphic representations. The other side contains terms such as orbital integrals. *Thanks to the trace formula, identities between orbital integrals on different groups imply identities between the representations of the two groups.*

It is possible to work backwards: from an analysis of the terms in the trace formula and a precise conjecture in representation theory, it is possible to make precise conjectures about identities of orbital integrals. The most

basic identity that appears in this way is the fundamental lemma, articulated above.

The proofs of many major theorems in automorphic representation theory depend in one way or another on the proof of a fundamental lemma. For example, the proof of Fermat's Last Theorem depends on Base Change for $GL(2)$, which in turn depends on the fundamental lemma for cyclic base change [17]. The proof of the local Langlands conjecture for $GL(n)$ depends on automorphic induction, which in turn depends on the fundamental lemma for $SL(n)$ [11], [12], [28]. Properties of the zeta function of Picard modular varieties depend on the fundamental lemma for $U(3)$ [26], [2]. Normally, the dependence of a major theorem on a particular lemma would not be noteworthy. It is only because the fundamental lemma has not been proved in general, and because the lack of proof has become a serious impediment to progress in the field, that the conjecture has become the subject of increased scrutiny.

7. REDUCTIONS

To give a trivial example of the fundamental lemma, if γ and γ_0 and their stable conjugates are not in any compact subgroup, then

$$\chi_{G,K}(g^{-1}\gamma'g) = 0 \text{ and } \chi_{H,K}(h^{-1}\gamma'h) = 0$$

so that both $\Lambda_{G,H}(\gamma)$ and $\Lambda_H^{st}(\gamma)$ are zero. Thus, the fundamental lemma holds for trivial reasons for such γ .

7.1. Topological Jordan decomposition. A somewhat less trivial reduction of the problem is provided by the topological Jordan decomposition. Suppose that γ lies in a compact subgroup. It can be written uniquely as a product

$$\gamma = \gamma_s \gamma_u = \gamma_u \gamma_s,$$

where γ_s has finite order, of order prime to the residue field characteristic p , and γ_u is topologically unipotent. That is,

$$\lim_{n \rightarrow \infty} \gamma_u^{p^n} = 1.$$

The limit is with respect to the p -adic topology. A special case of the topological Jordan decomposition $\gamma \in O_F^\times \subset \mathbb{G}_m(F)$ is treated in [13, p20]. In that case, γ_s is defined by the formula

$$\gamma_s = \lim_{n \rightarrow \infty} \gamma^{q^n}.$$

Let γ , γ_0 , and γ' be chosen as in Section 6.1. Each of these elements has a topological Jordan decomposition. Let $G_s = C_G(\gamma_{0s})$ and $H_s = C_H(\gamma_s)$. It turns out that G_s is an unramified reductive group with unramified endoscopic group H_s . Descent for orbital integrals gives the formulas [20] [8]

$$\Lambda_{G,H}(\gamma) = \Lambda_{G_s,H_s}(\gamma_u)$$

$$\Lambda_H^{st}(\gamma) = \Lambda_{H_s}^{st}(\gamma_u).$$

This reduces the fundamental lemma to the case that γ is a topologically unipotent elements.

7.2. Lie algebras. It is known (at least when the p -adic field F has characteristic zero), that the fundamental lemma holds for fields of arbitrary residual characteristic provided that it holds when the p -adic field has sufficiently large residual characteristic [9]. Thus, if we are willing to restrict our attention to fields of characteristic zero, we may assume that the residual characteristic of F is large. In fact, in our discussion of a reduction to Lie algebras in this section, we simply assume that the characteristic of F is zero.

A second reduction is based on Waldspurger's homogeneity results for classical groups. (Homogeneity results have since been reworked and extended to arbitrary reductive groups by DeBacker, again assuming mild restrictions on G and F .)

When the residual characteristic is sufficiently large, there is an exponential map from the Lie algebra to the group that has every topologically unipotent element in its image. Write

$$\gamma_u = \exp(X),$$

for some element X in the Lie algebra. We may then consider the behavior of orbital integrals along the curve $\exp(\lambda^2 X)$. A difficult result of Waldspurger for classical groups states that if $|\lambda| \leq 1$, then

$$\begin{aligned} \Lambda_{G,H}(\exp(\lambda^2 X)) &= \sum a_i |\lambda|^i \\ \Lambda_H^{st}(\exp(\lambda^2 X)) &= \sum b_i |\lambda|^i; \end{aligned}$$

that is, both sides of the fundamental lemma identity are polynomials in $|\lambda|$. If a polynomial identity holds when $|\lambda| < \epsilon$ for some $\epsilon > 0$, then it holds for all $|\lambda| \leq 1$. In particular, it holds at γ_u for $\lambda = 1$. The polynomial growth of orbital integrals makes it possible to prove the fundamental lemma in a small neighborhood of the identity element, and then conclude that it holds in general. In this manner, the fundamental lemma can be reduced to a conjectural identity in the Lie algebra.

8. THE PROBLEM OF BASE POINTS

The fundamental lemma was formulated above with one omission: we never made precise how to fix an isomorphism $T_H \leftrightarrow T_G$ between Cartan subgroups in H and G . Such isomorphisms exist, because the two Cartan subgroups have the same root data. But the statement of the fundamental lemma is sensitive to how an isomorphism is selected between T_H and a Cartan subgroup of G . If we change the isomorphism, we change the κ -orbital integral by a root of unity $\zeta \in \mathbb{C}^\times$. The correctly chosen isomorphism will depend on the element $\gamma \in H(F)$.

The ambiguity of isomorphism was removed by Langlands and Shelstad in [19]. They define a *transfer factor* $\Delta(\gamma_H, \gamma_G)$, which is a complex valued function on $H(F) \times G(F)$. The transfer factor can be defined to have the

property that it is zero unless $\gamma_H \in H(F)$ is strongly regular semisimple, $\gamma_G \in G(F)$ is strongly regular semisimple, and there exists an isomorphism (preserving character groups) from the centralizer of γ_H to the centralizer of γ_G . There exists $\gamma_0 \in G(F)$ such that

$$(8.0.1) \quad \Delta(\gamma_H, \gamma_0) = 1.$$

The correct formulation of the fundamental lemma is to pick the base point $\gamma_0 \in G(F)$ so that Condition 8.0.1 holds.

For classical groups, Waldspurger gives a simplified formula for the transfer factor Δ in [31]. Furthermore, because of the reduction of the fundamental lemma to the Lie algebra (Section 7.2), the transfer factor may be expressed as a function on the Lie algebras of G and H , rather than as a function on the group.

8.1. Base points for unitary groups. More recently, Laumon (while working on the fundamental lemma for unitary groups) observed a similarity between Waldspurger's simplified formula for the transfer factor and the explicit formula for differentials that is found in [27]. In this way, Laumon found a simple description of the matching condition $\gamma \leftrightarrow \gamma_0$ implicit in the statement of the fundamental lemma.

9. GEOMETRIC REFORMULATIONS OF THE FUNDAMENTAL LEMMA

From early on, those trying to prove the fundamental lemma have sought geometric interpretations of the identities of orbital integrals. Initially these geometric interpretations were rather crude. In the hands of Goresky, Kotwitz, MacPherson, and Laumon these geometric interpretations have become increasingly sophisticated. [5], [6], [21], [22].

This paper is intended to give an introduction to the fundamental lemma, and the papers giving a geometric interpretation of the fundamental lemma do not qualify as introductory material. In this section, we will be content to describe the geometric interpretation in broad terms.

9.1. Old-style geometric interpretations: buildings. We begin with a geometric interpretation of the fundamental lemma that was popular in the late seventies and early eighties. It was eventually discarded in favor of other approaches when the combinatorial difficulties became too great.

This approach is to use the geometry of the Bruhat-Tits building to understand orbital integrals. We illustrate the approach with the group $G = SL(2)$. The term $\chi_{G,K}(g^{-1}\gamma'g)$ that appears in the fundamental lemma can be manipulated as follows:

$$\begin{aligned} \chi_{G,K}(g^{-1}\gamma'g) \neq 0 &\Leftrightarrow g^{-1}\gamma'g \in K \\ &\Leftrightarrow \gamma'g \in gK \\ &\Leftrightarrow \gamma'(gK) = (gK) \\ &\Leftrightarrow gK \text{ is a fixed point of } \gamma' \text{ on } G(F)/K. \end{aligned}$$

The set $G(F)/K$ is in bijective correspondence with a set of vertices in the Bruhat-Tits building of $SL(2)$. Thus, we may interpret the orbital integral geometrically as the number of fixed points of γ' in the building that are vertices of a given type.

Under this interpretation, it is possible to use counting arguments to obtain explicit formulas for orbital integrals as a function of γ' . In this way, the fundamental lemma was directly verified for a few groups of small rank such as $SL(2)$ and $U(3)$.

9.2. Affine grassmannians. Until the end of Section 9, let $F = k((t))$, a field of formal Laurent series. Except for the discussion of the results of Kazhdan and Lusztig, the field k will be taken to be a finite field: $k = \mathbb{F}_q$.

In 1988, Kazhdan and Lusztig showed that if $F = \mathbb{C}((t))$, then $G(F)/K$ can be identified with the points of an ind-scheme (that is, an inductive limit of schemes) [15]. This ind-scheme is called the affine Grassmannian. The set of fixed points of an element γ can be identified with the set of points of a scheme over \mathbb{C} , known as the affine Springer fiber. The corresponding construction over $\mathbb{F}_q((t))$ is mentioned briefly in the final paragraphs of their paper. Rather than counting fixed points in the building, orbital integral can be computed by counting the number of points on a scheme over \mathbb{F}_q .

Based on a description of orbital integrals as the number of points on schemes over finite fields, Kottwitz, Goresky, and MacPherson give a geometrical formulation of the fundamental lemma. Furthermore, by making a thorough investigation of the equivariant cohomology of these schemes, they prove the geometrical conjecture when γ comes from an unramified Cartan subgroup [5].

9.3. Geometric interpretations. Each of the terms in the fundamental lemma has a nice geometric interpretation. Let us give a brief description of the geometrical counterpart of each term in the fundamental lemma. We work with the unitary group, so that we may include various insights of Laumon.

The geometrical counterpart of cosets gK are self-dual lattices in a vector space V over F .

The counterpart of the support set, $\text{SUP} = \{g \mid g^{-1}\gamma g \in K\}$, is the affine Springer fiber X_γ .

The counterpart of the integral of the support set SUP over G is counting points on the scheme X_γ . The integral over all of G diverges and the number of fixed points on the scheme is infinite. For that reason the orbital integral is an integral over $T \backslash G$, where T is the centralizer of γ , rather than over all of G .

The counterpart of the integral over $T \backslash G$ is counting points on a quotient space $Z_\gamma = X_\gamma / \mathbb{Z}^\ell$. (There is a free action of a group \mathbb{Z}^ℓ on X_γ , and Z_γ is the quotient.)

The geometric counterpart of $\kappa(a_\tau)$ is somewhat more involved. For elliptic endoscopic groups of unitary groups κ has order 2. The character κ

has the form.

$$\kappa : H^1(\mathrm{Gal}(\bar{F}/F), T) \cong (\mathbb{Z}/2\mathbb{Z})^\ell \rightarrow \{\pm 1\}.$$

The character κ pulls back to a character of \mathbb{Z}^ℓ . The rational points of X_γ are identified with self-dual lattices: $A^\perp = A$. The points of the quotient space Z_γ are lattices that are self-dual modulo the group action: $A^\perp = \lambda \cdot A$, for some $\lambda \in \mathbb{Z}^\ell$. The character κ then partitions the points of Z_γ into two sets, depending on the sign of $\kappa(\lambda)$:

$$Z_\gamma^\pm = \{A \mid A^\perp = \lambda A; \quad \kappa(\lambda) = \pm 1\}.$$

(In a more sophisticated treatment of $\kappa(a_\tau)$, it gives rise to a local system on Z_γ ; and counting points on varieties gives way to Grothendieck's trace formula.)

The counterpart of the κ -orbital integral $\Lambda_{G,H}(\gamma)$ is the number

$$\#Z_\gamma^+ - \#Z_\gamma^-.$$

The counterpart of the stable-orbital integral $\Lambda_H^{st}(\gamma)$ is the number

$$\#Z_\gamma^{H,st}$$

for a corresponding variety constructed from the endoscopic group.

The factors $\prod_\Phi |\alpha(\gamma) - 1|^{1/2}$ that appear on the two sides of the fundamental lemma can be combined into a single term

$$\prod_{\alpha \in \Phi_G \setminus \Phi_H} |\alpha(\gamma) - 1|^{1/2}.$$

This has the form q^{-d} for some value $d = d(\gamma)$. The factor q^{-d} has been interpreted in various ways. We mention that [24] interprets q^d as the points on an affine space of dimension d . That paper expresses the hope that it might be possible to find an embedding $Z_\gamma^- \rightarrow Z_\gamma^+$ such that the complement of the embedded Z_γ^- in Z_γ^+ is a rank d fiber bundle over $Z_\gamma^{H,st}$. The realization of this hope would give an entirely geometric interpretation of the fundamental lemma. Laumon and Rapoport found that this construction works over $\mathbb{F}_{q^2}((t))$, but not over $\mathbb{F}_q((t))$. In more recent work of Laumon, the constant d is interpreted geometrically as the intersection multiplicity of two singular curves.

9.4. Compactified Jacobians. Laumon, in the case of unitary groups, has made the splendid discovery that the orbital integrals – as they appear in the fundamental lemma – count points on the compactification of the Jacobians of a singular curve associated with the semisimple element γ . (In fact, Z_γ is homeomorphic to and can be replaced with the compactification of a Jacobian.) Thus, the fundamental lemma may be reformulated as a relation between the compactified Jacobians of these curves. By showing that the singular curve for the endoscopic group H is a perturbation of the singular curve for the group G , he is able relate the compactified Jacobians

of the two curves, and prove the fundamental lemma for unitary groups (assuming a purity hypothesis related to the cohomology of the schemes).

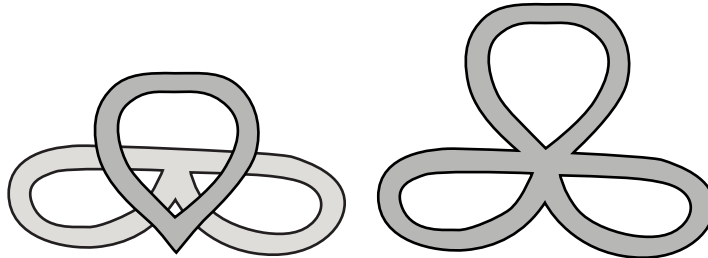


FIGURE 1. The singular curve on the left can be deformed into the singular curve on the right by pulling up on the center ring. The curve on the left controls $\Lambda_H^{st}(\gamma)$, and the curve on the right controls $\Lambda_{G,H}(\gamma)$. This deformation relating the two curves is a key part of Laumon's work on the fundamental lemma for unitary groups.

The origin of the curve C is the following. The ring $O_F[\gamma]$ is the completion at a point of the local ring of a curve C . In the interpretation in terms of Jacobians, the self-dual lattices $A^\perp = A$ that appear in the geometric interpretation above are replaced with O_C -modules, where O_C is the structure sheaf of C .

The audio recording of Laumon's lecture at the Fields Institute on this research is highly recommended [23].

9.5. Final remarks.

Remark 9.1. The fundamental lemma is an open ended problem, in the sense that as researchers develop new trace formulas (the symmetric space trace formula [14], the twisted trace formula [16], and so forth) and as they compare trace formulas for different groups, it will be necessary to formulate and prove generalized versions of the fundamental lemma. The version of the fundamental lemma stated in this paper should be viewed as a template that should be adapted according to an evolving context.

Remark 9.2. The methods of Goresky, Kottwitz, MacPherson, and Laumon are limited to fields of positive characteristic. This may at first seem to be a limitation of their method. However, there are ideas about how to use motivic integration to lift their results from positive characteristic to characteristic zero (see [3]). Waldspurger also has results about lifting to characteristic zero that were presented at the Labesse conference, but I have not seen a preprint [32].

Remark 9.3. In some cases, it is now known how to deduce stronger forms of the fundamental lemma from weaker versions. For example, it is known how to go from the characteristic function of the hyperspecial maximal compact

groups to the full Hecke algebra [9]. A descent argument replaces twisted orbital integrals by ordinary orbital integrals. However, relations between weighted orbital integrals remain a serious challenge.

Remark 9.4. There has been much research on the fundamental lemma that has not been discussed in detail in this paper, including other forms of the fundamental lemma. For just one example, see [25] for the fundamental lemma of Jacquet and Ye. Other helpful references include [18] and [30].

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